Lecture 10: Divisors

Note Title 9/22/2019

Definition: A Noetherian local ring A is regular if

① minimal # of generators of m = $\dim A$

TFAE ② $\dim_k m/m^2 = \dim A$, $k = A/m$ residue field

Theorem: $\dim_k m/m^2 \geq \dim A$

(*) X: Noetherian integral, separated scheme which is regular of codimension one.

every local ring of dimension one is regular.

ξ: generic point of X $\rightsquigarrow$ $K := \mathcal{O}_{\xi, X}$ is a field

function field of X

What are the elements in K?

Definition:
- $D \subseteq X$ is a prime divisor if it is an integral subscheme of codimension one.
- $\operatorname{Div} X := \left\{ \underbrace{\sum_{i \in I} a_i D_i}_{\text{Weil divisor}} \;\middle|\; D_i \text{ prime divisor of } X,\ a_i \in \mathbb{Z},\ |I| < \infty \right\}$

  $\cup$

  $\sum_{i \in I} a_i D_i$ is called effective if $a_i \geq 0$, $\forall i \in I$

D: prime divisor on X $\rightsquigarrow$ η: generic point of D

$\mathcal{O}_{\eta, X}$ is a discrete valuation ring in K

Krull dimension

$$\dim A := \max \{ n \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ chain of primes in } A \}$$

ex. $\dim k[x_1, \cdots, x_n] = n$

Theorem: k field, A = integral domain, finitely generated k-algebra

$$\underset{\text{prime}}{\mathfrak{p} \triangleleft A} \implies \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \dim A$$

Dually, X: topological space

$$\dim X := \max \{ n \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ irreducible closed subset of } X \}$$

$Z \subseteq X$ irreducible closed subset

$$\operatorname{codim}(Z, X) := \max \{ n \mid Z = Z_0 \subsetneq \cdots \subsetneq Z_n \quad " \quad \}$$

Theorem $\implies$ $\dim Z + \operatorname{codim}(Z, X) = \dim X$, if $X = \operatorname{Spec} A$

In particular, $\dim \mathcal{O}_{x,X} = \operatorname{codim}(\overline{\{x\}}, X)$, $x \in X$

Let v_D be the valuation corresponding to D.

$f \in K^*$, f has <u>zero</u> along D if $v_D(f) > 0$

<u>pole</u> " if $v_D(f) < 0$

Lemma 1: $v_D(f) = 0$ for finitely many prime divisor D.

pf: $U := \operatorname{Spec} A$ s.t f is <u>regular</u>

i.e $f = \frac{a}{b}$, $b \notin m_{\mathfrak{p}}$, $\forall \mathfrak{p} \in \operatorname{Spec} A$

X: Noetherian $\Longrightarrow$ only finitely prime divisors in $X \setminus U$

It suffices to show the lemma for $X = \operatorname{Spec} A$

$v_D(f) > 0 \Longrightarrow D \subseteq \underline{\{ \mathfrak{p} \in \operatorname{Spec} A \mid f_{\mathfrak{p}} \in m_{\mathfrak{p}} \}}$

Zariski closed

and thus has finitely many components

As a Corollary, $(f) := \sum_{D:\text{ prime divisor}} v_D(f) \cdot D \in \operatorname{Div} X$ principal divisor

$v_D(fg) = v_D(f) + v_D(g) \Longrightarrow (fg) = (f) + (g)$

Definition: • $D_1 \sim D_2$ if $D_1 = D_2 + (f)$ linear equivalent

• $\operatorname{Cl} X := \operatorname{Div} X / \sim$ divisor class group

Proposition 1: A Noetherian domain, $X = \operatorname{Spec} A$

$$A = \text{UFD} \iff X \text{ normal}, \ \operatorname{Cl} X = 0$$

ex. $X = \mathbb{A}^n_k$, then $\operatorname{Cl} X = 0$

Proposition 2: $X = \mathbb{P}^n_k$

$\cup\!\!\mid$

$H = \{x_0 = 0\}$ hyperplane

Define $\underset{\operatorname{Div} X}{\overset{\cap}{D = \sum_i n_i D_i}} \longmapsto \underset{\mathbb{Z}}{\overset{\cap}{\sum_i n_i \deg(D_i)}}$

then ① $\deg D = d$, $\Longrightarrow D \sim dH$

② $f^* \in K$, $\Longrightarrow \deg(f) = 0$

①+② $\Rightarrow$ ③ $\deg : \operatorname{Cl} X \xrightarrow{\simeq} \mathbb{Z}$

pf: prime divisor of $X \iff$ irreducible hypersurface

$\iff$ irreducible homogeneous polynomial in $k[x_0, \cdots, x_n]$

A : Noetherian integral domain

A is UFD iff every prime ideal of height one is principal.

② $f = \dfrac{g}{h} = \dfrac{g_1^{r_1} \cdots g_k^{r_k}}{h_1^{r_1'} \cdots g_\ell^{r_\ell'}}$, each g_i, h_i homog. irreducible

$k[x_0, \cdots, x_n]$ is a UFD

$$0 = \deg f = \deg g - \deg h = \sum_{i=1}^{k} r_i \deg g_i - \sum_{i=1}^{l} r_i' \deg h_i$$
$$= \deg (f)$$

① $\deg D = 0 \iff D = \sum_i a_i D_i$, D_i prime divisor $= \{\underline{f_i} = 0\}$ homog. polynomial
$$\sum_i a_i \deg f_i = d$$

$\dfrac{\prod_i f_i^{a_i}}{x_0^d}$ is homog. of degree $0 \in K^*$

then $D - dH = (\quad)$

Proposition 3: $Z \subsetneq X$ closed, $U = X \setminus Z$

① $\mathrm{Cl}\, X \twoheadrightarrow \mathrm{Cl}\, U$

$\sum a_i D_i \longmapsto \sum_i a_i (\underline{D_i \cap U})$ ignore those terms w/ $D_i \cap U = \phi$

② $\mathrm{codim}(Z, X) \geq 2$, then $\mathrm{Cl}\, X \xrightarrow{\cong} \mathrm{Cl}\, U$

③ Z irreducible of codim 1

then $\mathbb{Z} \to \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}\, U \longrightarrow 0$
$1 \longmapsto 1 \cdot Z$

pf: D prime divisor in X $\Rightarrow$ $D \cap U$ either empty or prime divisor

$(f) = \sum a_i D_i$. then $(f|_U) = \sum a_i (D_i \cap U)$

$\rightsquigarrow \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}\, U$

① D prime divisor in U $\Longrightarrow$ $\overline{D}$ prime divisor in X

$\therefore \quad \mathrm{Cl}\,X \twoheadrightarrow \mathrm{Cl}\,U$

② Changing X by a codim ≥ 2 subset doesn't change prime divisors & their valuations

③ $\mathrm{Ker}(\mathrm{Cl}\,X \to \mathrm{Cl}(U))$ consists of divisors support in Z

$\uparrow$ $\quad \because Z$ is irreducible

$\mathbb{Z}$

ex. D: irreducible deg d curve in $\mathbb{P}^2$

then $\mathrm{Cl}(\mathbb{P}^2 \setminus D) \cong \mathbb{Z}/d\mathbb{Z}$

$d=3 \quad \pi_1(\overset{X}{\overset{\shortparallel}{\mathbb{P}^2 \setminus E}}) \cong \mathbb{Z}_3 \xRightarrow{\text{universal coefficient}} H^2(X,\mathbb{Z}) \cong \mathbb{Z}_3$

X stein $\quad \underset{\substack{\shortparallel \\ 0}}{H^1(X,\mathcal{O})} \to \underset{\substack{\cong \\ \mathbb{Z}_3}}{H^1(X,\mathcal{O}_X^*)} \xrightarrow{\cong} \underset{\substack{\cong \\ \mathbb{Z}_3}}{H^2(X,\mathbb{Z})}$

Proposition 4. X satisfies $(*) \Longrightarrow X \times \mathbb{A}^1$ satisfies $(*)$

$\mathrm{Cl}(X \times \mathbb{A}^1) \cong \mathrm{Cl}\,X$

Cartier Divisors

X: scheme $\rightsquigarrow \mathcal{K}^*$ sheafification of

$$\mathcal{K}^*(U) = S(U)^{-1}\Gamma(U, \mathcal{O}_X)$$

$$S(U) := \left\{ s \in \Gamma(U, \mathcal{O}_X) \;\middle|\; \begin{array}{l} s_x \text{ not zero divisor} \\ \text{in } \mathcal{O}_x, \forall x \in U \end{array} \right\}$$

$\mathcal{O}^*$

Definition: An element of $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ is called a <u>Cartier divisor</u> on X

$$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$$

$$\rightsquigarrow \quad \Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$$

images are principal Cartier divisors

Proposition 5. X satisfies $(*)$ & all local rings are UFD

then $\quad \mathrm{Div}\, X \xrightarrow{\simeq} \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$

$\cup$

principal Weil divisor $\xleftrightarrow{1:1}$ principal Cartier divisor

pf: UFD $\Rightarrow$ integrally closed $\therefore X$ is normal

$\mathcal{K} =$ constant sheaf $\underset{\sim}{K}$ function field of X

$\{f_i\} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ Cartier divisor

i.e. $f_i \in \Gamma(U_i, \mathcal{K}^*/\mathcal{O}^*)$, $\{U_i\}$ open cover of X

$$v_D(\{f_i\}) := v_D(\tilde{f}_i) \quad \text{if} \quad U_i \cap D \neq \phi$$

any lifting of f_i in K^*, possibly after refine the open cover

$\|$

$v_D(\tilde{f}_j)$ if $U_i \cap U_j \neq \phi$

$$\rightsquigarrow \sum_D v_D(\{f_i\}) D \in \mathrm{Div}\, X$$

finite sum because the support is closed & X is Noetherian

Conversely, $\sum a_i D_i$ Weil divisor on $X \ni x$

A UFD is PID iff every prime is maximal.

$\rightsquigarrow D_x$ Weil divisor on $\operatorname{Spec} \mathcal{O}_x$
$$D_x = (f_x), \quad f_x \in K$$

then $\exists$ open neighborhood $\mathcal{U}_x \ni x$ s.t. $D_x = (f_x)$ on $\mathcal{U}_x$

Then $\{(U_x, f_x)\}$ gives a Cartier divisor.

If $f_x, f_{x'}$ defines the same divisor $D|_{\mathcal{U}_x \cap \mathcal{U}_{x'}}$ on $\mathcal{U}_x \cap \mathcal{U}_{x'}$

$$f_x / f_{x'} \in T(\mathcal{U}_x \cap \mathcal{U}_{x'}, \mathcal{O}_X)$$

Invertible Sheaves

Proposition 6: $\mathcal{L}_1, \mathcal{L}_2$ invertible sheaves on X

① $\mathcal{L}_1 \otimes \mathcal{L}_2$ invertible sheaves

② $\exists \mathcal{L}_1^{-1}$ invertible sheaves s.t. $\mathcal{L}_1 \otimes \mathcal{L}_1^{-1} \cong \mathcal{O}_X$

pf: ① $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X$

② $\mathcal{L}_1^{-1} = \mathcal{H}om(\mathcal{L}_1, \mathcal{O}_X)$, then

$\mathcal{H}om(\mathcal{L}_1 \otimes \mathcal{L}_2) \cong \mathcal{L}_1^{-1} \otimes \mathcal{L}_2$

$$\mathcal{L}_1^{-1} \otimes \mathcal{L}_1 \cong \mathcal{H}om(\mathcal{L}_1, \mathcal{L}_1) \cong \mathcal{O}_X$$

Definition: Picard group of X. $\mathrm{Pic}(X) = \{\text{invertible sheaves on } X\}/\sim$

w/ addition $\otimes$

$$H^0(X, \mathcal{K}^*) \longrightarrow H^0(X, \mathcal{K}^*/\mathcal{O}^*) \longrightarrow H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{K}^*)$$

$f \longmapsto (f)$ $\quad$ $H^1(X,\mathcal{K}^*) \overset{?}{=} 0$

Given a Cartier divisor $D = \{(U_i, f_i)\}$, $f_i \in \Gamma(U_i, \mathcal{K}^*)$

$f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ if $U_i \cap U_j \neq \phi$

Consider the $\mathcal{O}_X$-module generated by f_i^{-1} in $\mathcal{K}$

$\rightsquigarrow$ invertible sheaf $\mathcal{L}(D)$

$\{ f \in \mathcal{K} \mid D + (f) \geq 0 \}$

D viewed as Weil divisor

Proposition 7: X scheme

① $\{\text{Cartier divisor on } X\} \xrightarrow{1:1} \{\text{invertible subsheaf of } \mathcal{K}\}$

$D \longmapsto \mathcal{L}(D)$

② $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

③ $D_1 \sim D_2 \iff \mathcal{L}(D_1) \cong \mathcal{L}(D_2)$

pf: $\mathcal{O}_X|_{U_i} \xrightarrow{\cong} \mathcal{L}(D)|_{U_i}$ thus $\mathcal{L}(D)$ invertible

$1 \longmapsto f_i^{-1}$

① $\mathcal{K} \supseteq \mathcal{L}(D) \rightsquigarrow f_i^{-1} =$ generator of $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_{U_i}$

$\{f_i\}$ gives a Cartier divisor

② If $\mathcal{L}(D_1)$ locally generated by f_i^{-1} on U_i

$\mathcal{L}(D_2)$ " g_i^{-1} "

then $\mathcal{L}(D_1 - D_2)$ " $f_i^{-1} g_i$

$\mathcal{L}(D_1-D_2) \cong \mathcal{L}(D_1)\mathcal{L}(D_2)^{-1} \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

③ By ②, it suffices to prove that

$D \sim 0$ iff $D = (f)$, $f \in \Gamma(X, \mathcal{K}^*)$

Definition: $\mathrm{CaCl}(X) := \{\text{Cartier divisor on } X\} / \text{principal Cartier divisor}$

Theorem: X integral scheme, then $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$

pf: It suffices to prove that

every invertible sheaf of X is a subsheaf of $\mathcal{K}$

X integral $\Rightarrow \mathcal{K} \cong \underline{K}$ constant sheaf of $\underline{K}$

quotient field

$U \subseteq X$ open s.t $\mathcal{L}|_U \cong \mathcal{O}_U$

$$\mathcal{L} \otimes \mathcal{K}|_U \cong \mathcal{L}|_U \otimes \underline{K} \cong \mathcal{O}_U \otimes \underline{K} \cong \underline{K}$$

locally constant sheaf on an irreducible space is constant

$$\Longrightarrow \mathcal{L} \otimes \mathcal{K} \cong \underline{K}$$

$\uparrow \Leftarrow X$ integral

$\mathcal{L} \otimes \mathcal{O}_X = \mathcal{L}$

Linear System

X = non-singular projective variety$/_k$, $\mathcal{L}$ invertible sheaf

Proposition 4: D_0 divisor on X, $\mathcal{L} \cong \mathcal{L}(D_0)$ the corresponding invertible sheaf

① $\forall\, s \in \Gamma(X, \mathcal{L})$, $s \neq 0$, $(s) \sim D_0$, $s \geq 0$

② Every effective divisor, linear equivalent to D_0 is (s), for some $s \in \Gamma(X, \mathcal{L})$

③ $s, s' \in \Gamma(X, \mathcal{L})$, $(s) = (s')$ iff $\exists\, \lambda \in k^*$ $s = \lambda s'$.

pf: ① $\mathcal{L} \cong \mathcal{L}(D_0) \subseteq \mathcal{K}$,

$s \in \Gamma(X, \mathcal{L}) \longleftrightarrow f \in K^*$, $\underbrace{(f) + D_0}_{(s)} \geq 0$

Assume that $D_0 = \{(U_i, f_i)\}$, $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} f_i^{-1}$

② If $D = D_0 + (f)$, $f \in K^*$ & $D \geq 0$
$\Downarrow$
$f \in \Gamma(X, \mathcal{L}(D_0))$ $\Longleftarrow$ $(f) \geq -D_0$
w/ (f)

③ $s/s' \in K^*$ s.t $(s/s') = 0$ $\longrightarrow$ $s/s' \in \Gamma(X, \mathcal{O}^*) \cong k^*$
X: projective

$V \subseteq \Gamma(X, \mathcal{L})$
subspace finite dimensional

• $\sigma = V/k^*$ is called a linear system
$\updownarrow$
$\{D \in Div(X) \mid \underset{\overset{\vee\!/}{0}}{D} = (s),\ s \in \Gamma(X, \mathcal{L})\}$

• σ is a complete linear system if $V = \Gamma(X, \mathcal{L})$

• Base locus of σ is $\{p \in X \mid p \in \operatorname{Supp} D,\ \forall D \in \sigma\}$